APPROXIMATE ANALYTICAL SOLUTIONS TO STEFAN'S PROBLEMS

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of

MASTER OF TECHNOLOGY

By

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to the

DEPARTMENT OF MECHANICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

AUGUST, 1981

CERTIFICATE

This is to certify that the thesis entitled, 'Approximate Analytical Solution to Stefan Problems', by C.B. Paliwal is a record of work carried out under my supervision and has not been submitted elsewhere for a degree.

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ACKNOWLEDGEMENTS

I take pleasure in expressing my heartiest thanks and gratitude to my guide Dr.J.S.Goela whose able instructions and readiness to help enabled me to complete my thesis. I am thankful to Dr.J. Srinivasan for many important discussions I had with him. I sincerely and gratefully acknowledge Dr.Y. Jaluria for awakening my interest in this field.

My thanks are due to all of my friends who helped me in completing the work directly or indirectly.

Thanks are also due to Mr.U.S. Misra whose faultless typing made the task of proof reading pleasant and easy.

August 1981. -C.B. Paliwal

CONTENTS

NOMENCLATURE

AIO	Absorbed part of incident energy flux
C_p	Heat capacity
I_o	Incident energy flux
K	Thermal conductivity
L	Total length of the slab
L_m	Latent heat of fusion
nr	Space grid containing fusion front
Q	Heat flux prescribed
R(t)	Position of interface
r	Space coordinate in cylindrical case
T	Temperature
t	time
X(t)	Position of interface in semi-infinite slab
x	Space coordinate in semi_infinite case
α	Absorptivity of the material for the incident energy flux
γ	Diffusivity
θ	Nondimensional temperature
ρ	Density
τ_v	Time interval between onset of melting and vaporisation
Δr	Space grid size in cylindrical form
Δt	Time step size
Δx	Space grid size in plane case
λ	Fourier number = $\gamma t / x^2$

λ_c Coefficient in interface position and time relation for constant temperature case = $R(t)/2\sqrt{\alpha t}$

Subscripts

i	ith grid (space variable)
L	Liquid
m	melting
S	solid
V	vaporisation

Superscripts

*	Nondimensional variable
'	Variable for next time step

LIST OF FIGURES AND TABLES

ABSTRACT

In this work an attempt has been made to find a relatively simple approximate analytical solution to the Stefan's problems i.e. moving interface boundary problem in heat diffusion. An approximation has been made in the interface boundary condition which facilitates to give a simple analytical solution.

This model has been applied to first and second kind of boundary conditions . The model is applicable for a semi-infini slab. The onset of vaporisation time and the new phase zone width at this time has been found from the above model for different materials in case of second kind of boundary condition. A computer program has been developed to find the numerical solutions for the same problem without any approximation. The model's results have been compared with the computataonal results thus obtained and with other published results. Comparison shows that models's results agree with 5% with the actual results for the case of meltzone width at the time of onset of vaporisation. But time for the onset of vaporisation as given by model is as much as 30% off from the actual results for chromium. The effect of time varying heat flux on the time of onset of vaporisation and on the width of new phasezone at this time has been discussed.

The computer program as developed in the case of semi-infinite slab has been modified for the case of cylondrical geometry. Temperature distribution. new phasezone width and

the time of onset of vaporisation has been calculated for different materials in thecase of a hollow infinite cylonder (infinite in length and outer radius, while with finite inner radius) with second kind of boundary condition.

I. CHAPTER

INTRODUCTION:

Many practical heat transfer processes are associated with change of phase of material due to either melting or freezing. Applications include the welding of materials, solidification of castings and iceformation. Melting of solid by the application of heat flux is very important phenomenon in case of welding. In this application the depth of melted zone prior to vaporisation and time for onset of vaporisation are the main quantities of interest. In this study the emphasis has been placed on these two quantities.

The main feature of this type of problems is the existence of an interface which separates two regions of different thermophysical properties. This interface moves as some function of time. At the interface energy is either released or absorbed. Stefan ( ) first published his pioneering work on these type of problems and hence these problems are referred as "Stefan problems". One important feature of Stefan problem is the nonlinear nature of the interface boundary condition. Due to this, the solutions of diffusion equation cannot be superimposed and therefore particular solutions of the differential equation have to be obtained separately for each situation. For instance even if a solution satisfying the nonlinear interface condition is found, the same solution may not satisfy the flux boundary condition. To solve this problem many investigators have tried

approximations, which eliminate the use of diffusion equation in one particular phase. Goodman (1958) assumed the solid phase to be at the melting temperature and thus diffusion equation is to be applied only in liquid phase while Boley (1961) considered the case of ablation, eliminating liquid phase. But in both these cases all the thermophysical properties of one phase have been ignored. In this study also the use of diffusion in solid phase has been avoided but the density and heat capacity of the solid phase has been accounted for.

The results obtained under these assumptions have been compared with the analogue-computer results of Cohen (1967). Numerical solutions of the exact problem have also been obtained and compared with the results of Cohen (1967) and that of Hsu, Mehrabin and Chakraborty (1978). Furthermore, analytical results obtained from our model have been compared with our numerical results. Since exact analytical solutions exist for the constant temperature case, for comparison purposes, we have also compared our analytical results with them. In addition, the effect of time varying flux has been discussed, and numerical solutions have been obtained for the case of cylindrical geometry with heat flux boundary condition.

In Chapter II, literature pertinent to Stefan problem has been reviewed. Formulation of the exact problem and approximation proposed has been discussed in Chapter III. Method of the solution for these problems have been discussed

in Chapter IV. Numerical formulation of the problem and problems encountered in it has also been discussed in this chapter. Results are discussed in Chapter V.

II CHAPTER

Literature Review

The first published work on moving boundary problems in heat diffusion was that of Stefan. The original reference has been quoted by Carslaw and Jaeger (1). He obtained analytical solutions for temperature profile and the interface velocity for first and second kind of boundary conditions. In second kind of boundary condition, interface velocity was assumed constant. The initial temperature was assumed to be phase change temperature thus making it only one phase problem. First important exact solution was determined by Franz Neumann. The solutions has been given in reference (1). His solutions applied to a semi-infinite slab with the constant temperature boundary condition. Initial temperature of the slab was taken to be more than phase change temperature for the case of solidification. The solutions are in terms of error function and are difficult to use as a trial and error procedure if is to be applied.

Methods of solutions published till now are basically of threedifferent nature i.e. analutical, alalog computation and digital computation. Even G.W.(2) found the meltzone width in the form of Taylor series in time. The temperature distribution is also given in the form of Taylor series in terms of time and space variable. These solutions are valid for very small time period. Landau (3) formulated the governing equation in a very general form and gave analytical solutions for

effects. Goodman (4) has obtained some approximate solution using integral heat balance method and assuming a simpler form of temperature distribution in thechanged phase. The solutions obtained in this paper are applicabbe only when diffusion equation is to be applied in only one phase e.g. if material is initially at phase change temperature. In his future papers he considered some general cases also . Boley(5) has also solved the problem analytically but considering the case of ablation and thus diffusion equation is to be applied only in one phase.

The most complete analysis of numerical method was presented by Eyres in a pioneer paper. He developed lumped parameter system and solved on analog computer for seme-infinite case with body initially at phase change temperature. The reference of this paper and of Otis has been given by Muehlbauer, J.C.(6). Otis adopted the concept of moving heat source to account for the latent heat. Thismethod required a coordinate transformation in terms of an artificial time variable, which limits the analysis to materials initially at fusion temperature. After this an important work was that of Murray (7) in which he formulated the problem in finite difference form in two ways: one was of variable space network, in which grid size varies with time to track the interface position. Other uses fixed space network system, which has been used in this study also for numerical computation. Here grid size is kept constant but the boundary is tracked and the temperature of grid containing interface boundary is linearly intrapolated. The paper given results only for a freezing problem in which the cody has been assumed

Another important work in this field is of CohenM.I. (8) whose results will be discussed later in this dtudy. He solved the problem of melting of semi-infinite slab due to prescribed constant heat flux on alalog computer. Results have been given for differnt materials. He has given results assuming the preperties of materials to be same in both phases.

W.L. Heitz (9) has extended the variable space network method given by Murray so that the initialisation could be improved. Till new for initialisation some arbitrary new phase zone width was already assumed by authors. Heitz solved a freezing problem and at start he assumes all the sensible heat of subcooling is sonverted instantly to latest heat resulting in a solid thickness. This thicknews is then used to proceed further to solve diffusion equation numerically. C.Benacinq (10) has developed three time-level implicit scheme, which is unconditionally stalbe and convergent on the basis of sn analytical approach consisting of the approximation of the latent heat effect by a large heat capacity over a small temperature range. Although apparent heat capacity formulations have the advantage of being simple to program, the predicted phase change interface location advances in an unphysical oscillatory fashion. Hsu, Mehrabin and Chakraborty (11) have solved the problem of meling due to laser irradication numbercally. They used variable space network method. They discussed the effect of laser beam

intensity on the time of onset of vaporisation. Their results will be compared with results of this study. A brief summary of literature review is given in tabular form on the next page.

S.No.	Investigator	Type of boundary conditions	Method of Solution	Remark
1.	Stefan (1889)	a. Constant temp. b. Varying flux	a. Analytical b. Analytical	a. $T_o = T_m$ b. $T_o = T_m$; interface velocity assumed constant which gives varying heat flux
2.	Neumann (1860)	Constant temp.	Analytical	General Solution
3.	Evans G.W. (1950)	Both types of boundary conditions	Analytical	Solution inform of Taylor series. Not applicable for large times
4.	Landu (1950)	Both	Analytical	Assumption of infinite or negligible latent heat effects
5.	Goodman (1958)	Both	Analytical	$T_o = T_m$; Integral heat balance method with assuming parabolic temperature profile in liquid region
6.	Boley (1961)	Both	Analytical	Ablation was considered
7.	Otis (1956)	Both	Numerical	$T_o = T_m$ is required. Moving heat source assumption to account for latent heat effect.

Contd...

S.No.	Investigator	Type of Boundary conditions	Method of Solution	Remark
8.	Murray (1959)	Both	Finite difference formulation is given	Variable space network method and fixed space network method. Discussed
9.	Cohen (1967)	Flux condition	Analog computation	Properties of material taken constant in both phases
10.	Heitz (1970)	Constant temp.	Numerical	Improved method of initialisation with energy balance
11.	Bonacina (1973)	Constant temp.	Numerical	Approximating latent heat effects by large heat capacity. Interface advances in oscillatory fashion
12.	Hsu et.al. (1978)	Constant flux	Numerical	Variable space network method used

III. CHAPTER

Exact formulation for the Stefan problem of semi-infinite slab with different boundary conditions and for infinite hollow cylinder with heat flux boundary condition is discussed below.

3.1 Semi-infinite Slab

Consider a solid semi-infinite slab initially at a uniform temperature T_o which is assumed to be less than the melting temperature T_m. The geometry and the coordinate system used are shown in Figure 1(a). A heat flux $Q(t)$ is applied to its surface at $x = 0$.

The governing heat diffusion equation and boundary and initial conditions as given in Ref.(4) can be written as

$$\frac{\partial T(x,t)}{\partial t} = \alpha_s \frac{\partial^2 T(x,t)}{\partial x^2} \quad ; \; x \geq 0 \; t_m > t > 0 \tag{1}$$

$$-K_s \frac{\partial T(0,t)}{\partial x} = Q(t) \quad ; \; t_m > t > 0 \tag{2}$$

$$-K_s \frac{\partial T(\infty,t)}{\partial x} = 0 \quad t > 0 \tag{3}$$

$$T(x,0) = T_o \quad ; \; x \geq 0 \tag{4}$$

$$\frac{\partial T(x,t)}{\partial t} = \alpha_L \frac{\partial^2 T(x,t)}{\partial x^2} \quad ; \; X(t) > x > 0, \; t_v > t > t_m \tag{5}$$

$$-K_L \frac{\partial T(x,t)}{\partial x} = Q(t) \quad ; \; t_v > t > t_m \tag{6}$$

$$T(X(t),t) = T_m \quad ; \quad t_v > t > t_m \tag{7}$$

$$\frac{\partial T(x,t)}{\partial t} = \alpha_s \frac{\partial^2 T(x,t)}{\partial x^2} \; ; \; x > X(t), \; t_v > t > t_m \tag{8}$$

with interface condition

$$-K_L \frac{\partial T(x^-(t),t)}{\partial x} = -K_s \frac{\partial T(x^+(t),t)}{\partial x} + \rho L_m \frac{dX(t)}{dt} \tag{9}$$

The system of equations (1)-(9) fully describe the problem of melting of semi-infinite slab due to prescribed heat flux. In case of constant temperature boundary condition equations (2) and (6) will change to

$$T(0, t) = T_s \; ; \; t > 0 \tag{10}$$

3.2 Hollow Infinite Cylinder

Consider a hollow infinite solid cylinder of internal radius r_i at a uniform temperature T_o which is assumed to be less than melting temperature T_m. The geometry and coordinate system used has been shown in Fig.1(d). There is only the difference of geometry between the previous problem and in this one. The governing equations and boundary and initial conditions can be written in a similar way as follows:

$$\frac{\partial T(r,t)}{\partial t} = \alpha_s \left(\frac{\partial^2 T(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial T(r,t)}{\partial r} \right) ; \; r > r_i \; t_m > t > 0 \tag{11}$$

$$-K_s \frac{\partial T}{\partial r}(r_i,t) = Q(t) \; ; \; t_m > t > 0 \tag{12}$$

$$-K_s \frac{\partial T(\infty,t)}{\partial r} = 0 \quad ; \quad t > 0 \tag{13}$$

$$T(r,0) = T_o \quad ; \quad r \geq r_i \tag{14}$$

$$\frac{\partial T(r,t)}{\partial t} = \alpha_L \left(\frac{\partial^2 T(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial T(r,t)}{\partial r} \right), \quad r > r_i, t_v > t > t_m \tag{15}$$

$$- K_L \frac{\partial T(r_i,t)}{\partial r} = Q(t) \quad ; \quad t_v > t > t_m \tag{16}$$

$$T(R(t),t) = T_m \quad ; \quad t_v > t > t_m \tag{17}$$

$$\frac{\partial T(r,t)}{\partial t} = \alpha_s \left(\frac{\partial^2 r,t}{\partial r^2} + \frac{1}{r} \frac{\partial T(r,t)}{\partial r} \right); \quad r > R(t), t_v > t > t_m \tag{18}$$

with interface boundary condition

$$-K_L \frac{\partial T(R^-(t),t)}{\partial r} = -K_s \frac{\partial T(R^+(t),t)}{\partial r} + \rho L_m \frac{dR(t)}{dt} \tag{19}$$

Analytical solution of equations (1)-(9) is difficult to obtain because of the interdependence of diffusion equations (5) and (8) through a nonlinear boundary condition (9). Problem becomes more complicated due to the existence of heat flux boundary condition at the surface. Evans G.W. has given analytical solution to this problem in the form of Taylor series, but solution can not be applied for large times as the series diverges. Other attempts e.g. by Goodman and Boley simplify the problem to one phase by suitable assumptions which are discussed in Chapter II. In all these attempts all the properties of the solid region have been ignored. In this thesis an attempt has been made to decouple the diffusion equation in solid region from the rest of the system. The

interface condition (9) is modified such that density and specific heat of solid is taken into account but the diffusion equation is not required to obtain melt zone width and the time for onset of vaporisation as discussed below.

3.3 Proposed Model

Consider the interface boundary condition (9). The left hand term denotes the heat flux coming from the liquid region and the right hand side terms denote heating and melting of solid. The proposed model is obtained as follows. Consider the grid system as shown in Fig.1(b). Just at the onset of melting we can write

$$-K_s \left.\frac{\partial T}{\partial x}\right|_{0^+} = C_p (T_m - T_1) \left.\frac{dX}{dt}\right|_1 - K_s \left.\frac{\partial T}{\partial x}\right|_{1^+} \qquad (20)$$

$$= C_p (T_m - T_1) \left.\frac{dX}{dt}\right|_1 + C_p (T_1 - T_2) \left.\frac{dX}{dt}\right|_2 - K_s \left.\frac{\partial T}{\partial x}\right|_{2^+} \qquad (21)$$

in a similar way

$$= C_p (T_m - T_1) \left.\frac{dX}{dt}\right|_1 + \cdots + C_p (T_n - T_{n+1}) \left.\frac{dX}{dt}\right|_{n+1} + C_p (T_{n+1} - T_{n+2}) \left.\frac{dX}{dt}\right|_{n+\frac{1}{2}} - K_s \left.\frac{\partial T}{\partial x}\right|_{\infty} \qquad (22)$$

but

$$K_s \left.\frac{\partial T}{\partial x}\right|_{\infty} = 0 \text{ and } T(\infty, t) = T_o \qquad (23)$$

Hence assuming $\frac{dX}{dt}$ is same for all grid points we can write

$$- K_s \frac{\partial T}{\partial x} (X^+(t),t) = C_p (T_m - T_o) \frac{dX(t)}{dt} \qquad (24)$$

Thus the condition (19) changes to

$$-K_L \frac{\partial T}{\partial x} (X^-(t),t) = \rho C_p (T_m - T_o) \frac{dX(t)}{dt} + \rho L_m \frac{dX(t)}{dt} \qquad (25)$$

$$-K_L \frac{\partial T(X^-(t),t)}{\partial x} = \rho L_m' \frac{dX(t)}{dt} \qquad (26)$$

where

$$L_m' = L_m + C_p (T_m - T_o) \qquad (27)$$

L_m', henceforth will be referred as modified latent heat of fusion.

IV. CHAPTER

Method of Solution

In this study the Stefan problem is solved in order to find the melt zone width and time for onset of vaporisation for different materials. Therefore calculations have been made to find $AI_o^2 \tau_V$ and $AI_o X_V$, where AI_o is the absorbed flux and τ_V and X_V are time between onset of melting and vaporisation and melt zone width at the onset of vaporisation respectively. These values have been found to be constant for a particular material. In the calculations made here, the properties of the materials have been averaged out in the temperature range concerned e.g. in case of solid the temperature concerned is from T_o to T_m while for liquid it is T_m to T_V.

Model

To find the analytical solution of the problem of semi-infinite slab with heat flux boundary condition equation (1)-(8) are to be solved with the modified boundary condition (26). This problem is now broken in two segments. One upto $t = t_m$, of which the standard solution as given in Ref.(4) is

$$T(x,t) = \frac{Q}{K_s} \frac{x}{y} \left(\frac{e^{-y^2}}{\sqrt{\pi}} - y \operatorname{erfc} y \right) \qquad (28)$$

where

$$y = x/(2\sqrt{\kappa_s t}); \quad \operatorname{erfc} y = \frac{2}{\sqrt{\pi}} \int_y^{\infty} e^{-z^2} dz \qquad (29)$$

Here Q is assumed to be a constant heat flux.

After the onset of melting two phase region exists. To simplify the solution without much loss of accuracy a parabolic distribution is assumed in the liquid region as follows:

$$T(x,t) = A(x-X(t)) + B(x - X(t))^2 + T_m \qquad (30)$$

Thus now the solution obtained by Goodman can be applied here. The results are being quoted here from Ref.(1).

$$A = \frac{1}{2}\left[1 - (1 + 4\mu)^{1/2}\right] \frac{L_m'}{C_{pL}\, X(t)} \qquad (31)$$

$$B = \frac{1}{8} - \frac{L_m'}{C_{pL}\, X^2(t)}\left[1 - (1 + 4\mu)^{1/2}\right]^2 \qquad (32)$$

where

$$\mu = \frac{Q(t)\, X(t)}{k_L\, \rho L_m'} \qquad (33)$$

and

$$\frac{Q(t) \int_0^{\tau} Q(t)dt}{L \rho^2 L_m'^2} = \frac{\mu}{6}\left[\mu + 5 + (1 + 4\mu)^{1/2}\right] \qquad (34)$$

Surface temperature can be obtained from (30) by putting $x = 0$ and values of A and B. This gives

$$T_s - T_m = \frac{L_m'}{C_{pL}}\left[-\frac{1}{4} + \frac{1}{4}(1 + 4\mu)^{1/2} + \frac{\mu}{2}\right] \qquad (35)$$

Thus the solution obtained by Goodman for the problem of initial temperature $T_o = T_m$ can be applied even if T_o is less than T_m, by simply modifying the latent heat of diffusion as shown in eq.(27).

To verify the validity of the modified latent heat assumption, this simplification has also been applied for the case of constant surface temperature. Exact analytical solution as given by Neumannare being quoted here for comparison from Ref.(4).

$$T(x,t) = T_s - \frac{(T_s - T_m)}{\operatorname{erf}\lambda_c} \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_L t}}\right); \quad X(t) > x > 0 \tag{36}$$

$$T(x,t) = T_o - \frac{(T_o - T_m)}{\operatorname{erfc}\lambda_c\left(\frac{\alpha_L}{\alpha_s}\right)^{\frac{1}{2}}} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha_s t}}\right); \quad x > X(t) \tag{37}$$

where λ is given by

$$\frac{(T_s - T_m)e^{-\lambda_c^2}}{\operatorname{erf}\lambda_c} = \frac{K_s}{K_L}\left(\frac{\alpha_L}{\alpha_s}\right)^{\frac{1}{2}} \frac{(T_m - T_o)e^{-\lambda_c^2 \frac{\alpha_L}{\alpha_s}}}{\operatorname{erfc}\left(\lambda_c\left(\frac{\alpha_L}{\alpha_s}\right)^{\frac{1}{2}}\right)} + \pi^{\frac{1}{2}} \frac{L_m \lambda_c}{C_{pL}} \tag{38}$$

and interface position is given by

$$X(t) = 2\lambda_c (\alpha_L t)^{\frac{1}{2}} \tag{39}$$

While with the use of the proposed model, equation (9) changes to equation (26) and thus the solution can be obtained in a similar way as obtained above by Neumann. In this case equation (38) will change to

$$\frac{(T_s - T_m)e^{-\lambda_c'^2}}{\operatorname{erf}\lambda_c'} = \pi^{\frac{1}{2}} \frac{L_m' \lambda_c'}{C_{pL}} \tag{40}$$

which is in much simpler form. Other equations will remain same except the value of λ_c changes to λ_c' as obtained from (40).

Numerical Scheme

To solve the exact problem given by the set of equations (1)-(9) numerical computation has been carried out, so that the analytical results obtained from the model can be compared. Three point central difference formulation has been used in this scheme so that algebraic equations obtained, due to the implicit time scheme used, are in tridiagonal form and are much easier to solve. Figure 1(b) and (c) shows the fixed space network used. This procedure has been taken from Murray but he used parabolic interpolation to track the interface while here for the sack of simplicity without much loss of accuracy linear interpolation has been done. Other method given by Murray is of variable space network. In this tracking of interface is not difficult but the size of the grid varies with time. Implicit technique has been used to avoid the limitation on time or space grid size. In case of explicit method which is easy to compute this limitation can not be avoided because otherwise instability will arise in the solution.

First of all the variables have been non-dimensionalised as follows:

Subscripts being dropped where they are obvious from the context.

$$\theta = \frac{T - T_m}{T_v - T_m} \; ; \quad x^* = x/L \; ; \quad t^* = \frac{\alpha t}{L^2} \tag{41}$$

diffusion equation becomes

$$\partial\theta/\partial t^* = \partial^2\theta/\partial x^{*2} \tag{42}$$

in finite difference form:

$$-\lambda\theta'_{i-1} + (1 + 2\lambda)\,\theta'_i - \lambda\theta'_{i+1} = \theta_i \tag{43}$$

where

$$\lambda = \Delta t^*/\Delta x^{*2} \tag{44}$$

The space grids have been chosen as shown in Fig.1(b). Since first grid has been taken of full grid size the boundary condition (2) changes to

$$Q(t) = -K\frac{\partial T}{\partial x} + \rho C_p \Delta x \frac{\partial T}{\partial t} \tag{45}$$

in non-dimensional form it can be written as

$$-\frac{\partial\theta}{\partial x^*} + \Delta x^* \frac{\partial\theta}{\partial t^*} = \frac{Q(t)L}{K(T_V - T_m)} \tag{46}$$

and in finite difference form

$$(1 + \lambda)\,\theta'_1 - \lambda\theta'_2 = \theta_1 + \frac{Q(t)L\Delta t^*}{\Delta x^* K(T_V - T_m)} \tag{47}$$

other boundary conditions (3) and (7) can be written as

$$\theta_{N-1} - \theta_N = 0 \tag{48}$$

$$\theta(X(t)) = 0 \tag{49}$$

The moving interface boundary condition (9) in non-dimensional form becomes

$$V^* = -\left.\frac{\partial\theta}{\partial x^*}\right|_{x=X^-(t)} + \frac{K_s}{K_L}\left.\frac{\partial\theta}{\partial x^*}\right|_{x=X^+(t)} \tag{50}$$

where

$$v^* = v/v_c \; ; \quad v_c = \frac{(T_v - T_m)K_L}{\rho L_m L} \tag{51}$$

and

$$v = \frac{dX(t)}{dt} \tag{52}$$

In this scheme fixed space network method has been used hence it is not necessary that interface always lies on same grid point. Thus the interface is to be tracked at each time step and the temperature of the grid containing it is interpolated linearly. Thus if x is the distance of the interface from the grid point then equation (43) can be written in finite difference form as follows

$$v^*(t) = \frac{K_s}{K_L} \frac{\theta(nr+1)}{(\Delta x^* - \delta x^*)} + \frac{\theta(nr-1)}{(\Delta x^* + \delta x^*)} \tag{53}$$

$$; \quad \frac{\Delta x}{2} \geq \delta x \geq 0$$

In cylindrical geometry case the definition of variables changes as

$$r^* = r/(r_o - r_i); \; t^* = \frac{\alpha t}{(r_o - r_i)^2} \tag{54}$$

Diffusion equation (11) in nondimensional form is given as

$$\frac{\partial \theta}{\partial t^*} = \frac{\partial^2 \theta}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial \theta}{\partial r^*} \tag{55}$$

which in finite difference form reduces to

$$-\lambda\theta'_{i-1} + (1 + 2\lambda + \lambda \frac{\Delta r^*}{r^*})\theta'_i - (\lambda + \lambda \frac{\Delta r^*}{r^*})\theta'_{i+1} = \theta_i \tag{56}$$

where

$$\lambda = \Delta t^*/\Delta r^{*2} \tag{57}$$

Rest of the formation is same as in case of semi-infinite slab.

Initialisation and Tracking of Interface

In the case of numerical computation till the onset of melting there is no problem in computing temperature distribution for different time steps. As soon as the surface reaches the melting temperature, two phase problem is to be considered. Since at the start there is no liquid zone, the diffusion equation can not be applied there. Therefore first the velocity of interface, which is initially at $x = 0$ is found from the boundary condition (9). It is given as

$$V(t) = \frac{Q(t)}{\rho L_m} + \frac{K_s}{\rho L_m} \left.\frac{\partial T(t)}{\partial x}\right|_{x=X^+(t)} \qquad (58)$$

because

$$-K_L \frac{\partial T}{\partial x} = Q(t) \quad \text{at } x = 0 \qquad (59)$$

This equation is applied till the first grid is covered by liquid region. After this the temperature of the first grid point is found by taking energy balance for it, as given by equation (47). The temperature of the second grid point is interpolated between the first grid point and the interface temperature if it is in liquid region and between the third grid point and interface temperature if it is in solid region. This procedure is repeated till three grid points are covered by liquid region. After this the diffusion equation can be applied in liquid region as it exceeds three points.

The temperature of last point in the liquid region and first point in the solid region is found by linear interpolation between interface and the previous or next point respectively.

Main difficulty in the numerical computation comes in calculating the velocity of interface from eq.(53). This is because if the time step is large and the grid size is very small then in a single iteration the interface may cross more than one grid point. Since the temperature of the solid region was quite low and if we calculate slop in liquid region at boundary from this position it will come quite low and thus may result in negative velocity. Consequently, the time step is to be reduced or the grid size is to be increased. Thus even though the numerical scheme used here is in implicit form there is a limit on time or grid size to avoid this unphysical oscillatry motion of interface boundary. Moreover chosing Δx larger may result in higher error in computation.

Error Analysis

There are two types of errors in numerical computation

(i) Error due to finite difference formulation

(ii) Truncation error

In finite difference formulation of the diffusion equation the error will be of the order of $O(\Delta t) + O(\Delta x)^2$ which for the present calculations comes out to be less than 2%. Truncation error arise due to the finite number of significant digit calculations. Since in the calculation eight significant digits have been taken by the computer that itself is quite

accurate and if there is any error at the last digit place it is almost insignificant in comparison to the error made due to finite difference formulation.

V. CHAPTER

RESULTS AND DISCUSSION

First we discuss the results for the case of semi-infinite slab. Figs.(2) and (3) and Table (1-a) show a comparison between our analytical results, numerical results and the results of Cohen. Since Cohen obtained these results assuming same properties in both phases, for comparison purposes in Figs.(2) and (3) and Table (1-a) we have also done the same. It can be seen from these figures and table that the values of $AI_o X_V$ and $AI_o^2 \tilde{\tau}_V$ are constant for a given material. This result is well known and was also obtained by Cohen. From Fig.(2) we see that values of $AI_o X_V$ obtained from our analytical model agree within 5% with those of Cohen and our numerical results. We also see that the error in model results is positive for some materials and negative for others. Thus there is no definite trend and it is felt that the difference in results is so small that the positive and negative errors may occur due to inherent error in finite difference formulation.

The values of $AI_o^2 \tau_V$ obtained from our model differ as much as by 30% (for instance in the case of chromium) from the numerical values. For most of the materials discussed this difference is about 20%. Error percentage in the values of $AI_o X_V$ and $AI_o^2 \tau_V$ are given in Table (1-b) for different materials. From this table we see that the error

percentage in the case of $AI_o^2\tau_v$ is generally higher for those materials which have smaller values of thermal diffusivity, but some exceptions to this effect can also be seen. For instance, in the case of silver though diffusivity value is comparatively large, the error is also large. Similarly in the case of Platinum diffusivity value is comparatively low but the error is also low.

We have also obtained results for those situations in which the thermal properties of two phases are different. These results are given in Figs.(4) and (5) and Table (2). It can be seen that these results differ significantly (as much as by 155% for Gold) from those obtained by assuming constant properties in both phases. Therefore to obtain realistic results, it is necessary to take into account different properties in different phases.

Hsu et.al. have also obtained numerical results considering different properties in the two phases. In Fig.(4) we compare our numerical results with these of Hsu et.al. Even though Hsu et.al. used variable space network scheme while we used constant space network scheme, within 3% the two results are the same. Figs.(6) and (7) give the variation of X_v and τ_v with respect to absorbed flux for different materials. The results show that $x_v \sim I_o^{-1}$ and $\tau_v \sim I_o^{-2}$ as discussed above.

In Figs.(8) and (9) we show the effect of the flux-pulse-shape variation on τ_v and X_v. Two shapes one rectangular and the other Gaussian have been considered. Here the maximum

intensity for the two cases has been taken as $10^7 W/cm^2$. In comparison to the flat topped pulse, the Gaussian pulse takes more time to get the material upto the vaporisation stage. Further the melt zone width is also more with the Gaussian pulse. But looking from the energy point of view the rectangular pulse takes 285.76 J/cm^2 while the Gaussian pulse takes 744.36 J/cm^2 to reach upto the vaporisation stage. This is 2.6 times more than in the case of flat topped pulse, while the width melted is only 2.22 times more. Thus we see that if the quantity of interest is the melt zone width per unit of available energy than the rectangular pulse is better.

Table (3) gives the values of λ_c for the case of constant temperature boundary condition. Both these results have been obtained analytically, one with the proposed approximation and the other without any approximation as suggested by Neumann. The difference is more than 50% in most of the cases which shows that the model is not applicable for this case. One reason for this difference is that in this case there is no concept of melted thickness at the time of onset of vaporisation.

We have also obtained results for the case of cylindrical geometry numerically. These results are shown in Fig.(10). From this figure it can be seen that even for r_i as small as 0.15 cm the values are very near to that of semi-infinite slab case for Aluminium. This shows that analytical results obtained for the case of semi-infinite slab can be applied here

if the internal radius is relatively large. In general the numerical calculations show that if the internal radius is more that approximately ten times the melt zone width then the values obtained are very close to that of semi-infinite slab case.

CONCLUSIONS

In this study an attempt has been made to find a relatively simple analytical solution to Stefan problems i.e. moving interface problems in heat diffusion. An approximation was made in interface boundary condition to delink the liquid region from the solid region such that heat capacity and density of the solid phase is still accounted for. In the case of constant heat flux boundary condition the melt zone width at the time of onset of vaporisation obtained from the model is found to be in good agreement with the exact results obtained numerically and analog computer results obtained by Cohen. Percentage error in the values of $AI_o X_V$ is less than 5% but in the case of $AI_o^2 \tau_V$ it is sometimes as high as 30% as in the case of Chromium. But generally this error is around 20% for most of the other metals. In the case of constant temperature boundary condition, however, results obtained from the model differ from the actual analytical results by more than 50%. Numerical results have been obtained for the case of cylindrical geometry with constant heat flux

boundary condition. Results show that if the inner radius of the hollow cylinder is more than approximately ten times the melt zone width then the results of semi-infinite slab case can be applied. Concerning the improvement to the above model it is desirable that the thermal conductivity of the solid is taken into account as this property has not been included in the above model.

TABLE 1(a)

For Semi-infinite solid with constant heat flux boundary condition

* Materials	$AI_o^2.\tau_v$ $J^2/sec.cm^4$			$AI_o.X_v$ W/cm		
	Model's results	Nr.Computational results	Cohen's results	Model's results	Nr.Computational results	Cohen's results
Chromium	4.44	6.35	4.70	0.47	0.49	0.423
Copper	59.88	73.70	70.12	7.50	7.44	6.835
Gold	44.60	53.70	52.13	7.35	7.37	6.800
Iron	10.20	12.80	13.90	1.08	1.07	1.000
Nickel	14.90	19.00	19.48	1.40	1.41	1.256
Platinum	26.00	31.00	32.14	2.37	2.38	2.214
Rhenium	39.50	50.50	55.60	2.36	2.32	2.218
Silver	31.30	39.20	36.90	6.55	6.50	6.050
Tantalum	18.00	23.00	24.80	1.48	1.46	1.372
Tungston	60.00	79.60	84.50	3.87	3.79	3.745
Aluminium	29.60	34.40	-	5.77	5.94	-

* Thermo-physical properties of materials have been taken same in both phases and that of solid phase as given in Table (5).

- Result's not available.

TABLE 1(b)

*Materials / **Error Percentage	Cr	Cu	Au	Fe	Ni	Pt	Re	Ag	Tl	W	Al
in $AI_o^2\tau_v$	30.0	19.15	16.9	20.3	21.57	16.12	21.78	20.153	21.▼4	24.62	13.95
in AI_oX_v	4.08	-0.8	0.27	-0.93	0.71	0.42	-1.72	-0.77	-1.37	-2.0	2.86
Diffusivity cm^2//sec.	0.203	1.14	1.177	0.2125	0.2342	0.2468	0.2431	1.705	0.2362	0.4998	0.81

*Properties being same in both phases.

**Error percentage $= \frac{\text{Numerically obtained result - Model's result}}{\text{Numerically obtained results}} \times 100.$

TABLE 2

For semi-infinite solid with constant heat flux boundary condition

*Materials	$AI_o^2 \tau_v \times 10^{-6}$ $J^2/cm^4 sec.$		$AI_o X_v \times 10^{-3}$ W/cm.		$AI_o^2 t_m \times 10^{-6}$ $J^2/cm^4 sec.$
	Model's results	Nr. Computational results	Model's results	Nr. Computational results	
Chromium	3.44	4.79	0.31	0.30	5.725
Copper	28.00	38.6	3.46	3.38	11.900
Gold	18.60	25.0	2.93	2.88	6.270
Iron	6.70	9.55	0.62	0.66	4.800
Nickel	14.50	18.60	1.37	1.38	5.735
Platinum	62.60	70.00	5.22	5.32	4.886
Rhenium	54.00	68.30	2.57	2.51	16.050
Silver	15.00	20.00	2.90	2.83	7.010
Tantalum	21.00	26.80	1.68	1.66	8.590
Tungston	40.80	59.00	2.43	2.35	30.000
Aluminium	14.00	17.20	2.55	2.59	2.050

*Thermophysical properties of materials are taken different in two phases and as given in Table (5).

TABLE 3

For the case of semi-infinite solid with constant heat flux boundary condition

*Materials	$AI_o\sqrt{t_v} \times 10^{-3}$ J/sec$^{\frac{1}{2}}$cm^2		% difference between the two results
	Results of Ref.No.(11)	Nr.Computational results	
Aluminium	4.21	4.33	2.85
Iron	4.07	4.02	1.24
Nickel	4.27	4.16	2.644

*Properties of materials taken are given in Table (6).

TABLE 4

For the case of semi-infinite solid with constant surface temperature ($=T_B$) boundary condition

*Materials	Model's result λ_c^{**}	Exact analytical results λ_c^{**}
Chromium	0.895	0.53
Copper	0.910	0.63
Gold	1.01	0.74
Iron	0.952	0.62
Nickel	0.77	0.50
Platinum	1.01	0.70
Rhenium	1.045	0.70
Silver	0.96	0.655
Tantalum	0.837	0.53
Tungston	0.888	0.56
Aluminium	1.09	0.81

*Properties of materials are taken as given in Table (5).

** λ is non-dimensional variable given as

$$= \frac{X(t)}{2\sqrt{\alpha_L t}}$$

TABLE 5

Materials	ρ^* gm/cm^3	K_s W/cm°K	C_s J/gm°K	K_L W/cm°K	C_L J/gm°K	T_m °K	T_b °K	L_m J/gm.
Chromium (Cr)	7.176	0.67	0.46	0.4°	1.25°	2116.33	2755.0	339.6
Copper (Cu)	8.97	3.94	0.385	1.8	0.418°	1355.77	2866.33	211.65
Gold (Au)	19.35	2.96	0.130	1.16	0.15°	1355.77	3244.0	67.45
Iron (Fe)	7.85	0.754	0.452	0.44	0.78°	1812.0	3011.0	272.12
Nickel (Ni)	8.89	0.916	0.44	0.9°	0.44°	1728.0	3055.0	309.33
Platinum (Pt)	21.465	0.71	0.134	1.5°	0.18°	2046.3	4683.0	113.96
Rhenium (Re)	21.16	0.71	0.138	0.7°	0.30°	3440.0	6200.0	176.76
Silver (Ag)	10.476	4.18	0.234	1.8°	0.30°	1233.55	2483.0	104.66
Tantalum (Tl)	16.627	0.542	0.138	0.8°	0.167°	3266.33	5572.0	167.46
Tungston (W)	19.35	1.296	0.134	0.78	0.20°	3672.0	6200.0	183.74
Aluminium (Al)	2.7	2.3	1.05	1.0	1.2	933.0	2767.0	397.0

*Density has been taken same in both phases.

°Extrapolated values.

Thermophysical properties value given here are average values in the temperature range concerned i.e. for solid from room temperature upto melting temperature and for liquid from melting temperature to vaporisation temperature.

TABLE 6

Materials	ρ gm/cm^3	K_s W/cm°K	C_s J/gm°K	K_L W/cm°K	C_L J/gm°K	T_m °K	T_b °K	L_m J/gm.
Aluminium(Al)	2.7	2.28	1.048	1.08	1.086	933.0	2728.0	395.0
Iron (Fe)	7.86	0.42	0.691	0.44	0.628	1808.0	3273.0	272.0
Nickel (Ni)	8.9	0.74	0.556	0.43	0.656	1728.0	3005.0	299.0

Above properties has been taken by Hsu, Mehrabin and Chakravorty (1978) in their calculations.

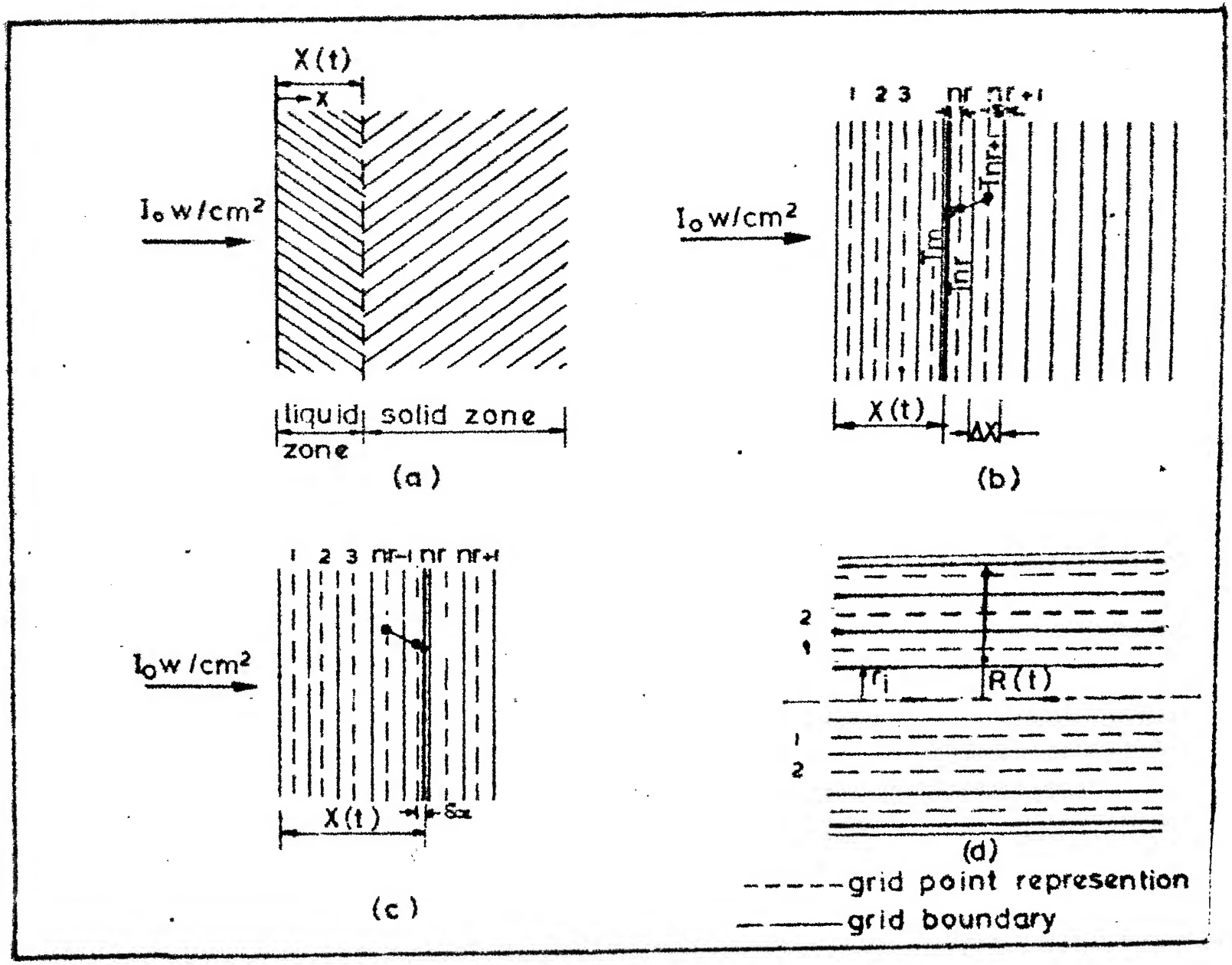
X(t)
X
I_o w/cm²
liquid zone
solid zone
(a)
1 2 3 nr nr+1
I_o w/cm²
Tm
Tnr
Tnr+1
X(t)
ΔX
(b)
1 2 3 nr-1 nr nr+1
I_o w/cm²
X(t)
δx
(c)
2
1
r_i
R(t)
1
2
(d)
-----grid point represention
———grid boundary

FIG. 1

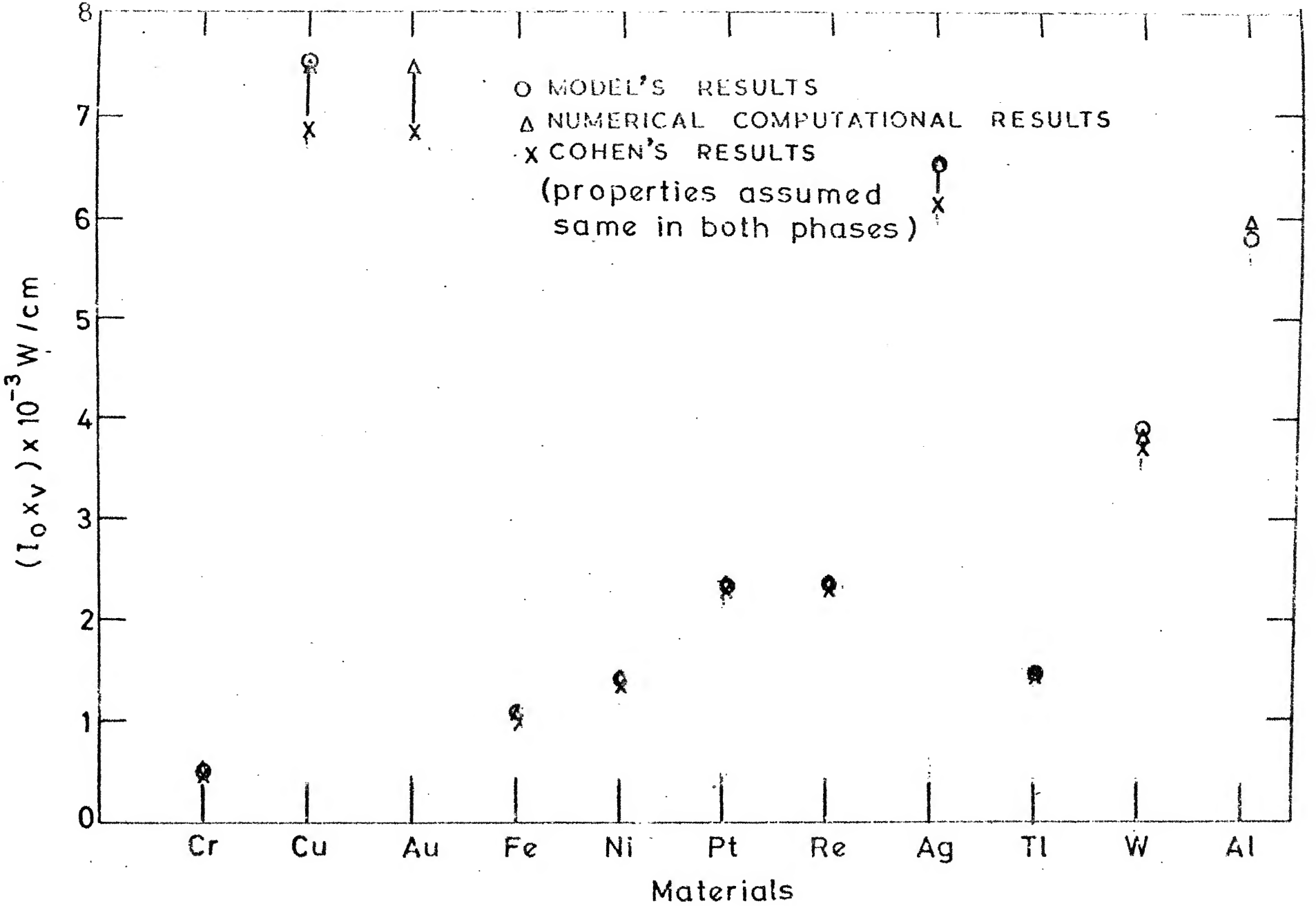
O MODEL'S RESULTS
Δ NUMERICAL COMPUTATIONAL RESULTS
X COHEN'S RESULTS
(properties assumed same in both phases)
$(I_o x_v) \times 10^{-3}$ W/cm
0
1
2
3
4
5
6
7
8
Cr
Cu
Au
Fe
Ni
Pt
Re
Ag
Tl
W
Al
Materials

FIG. 2

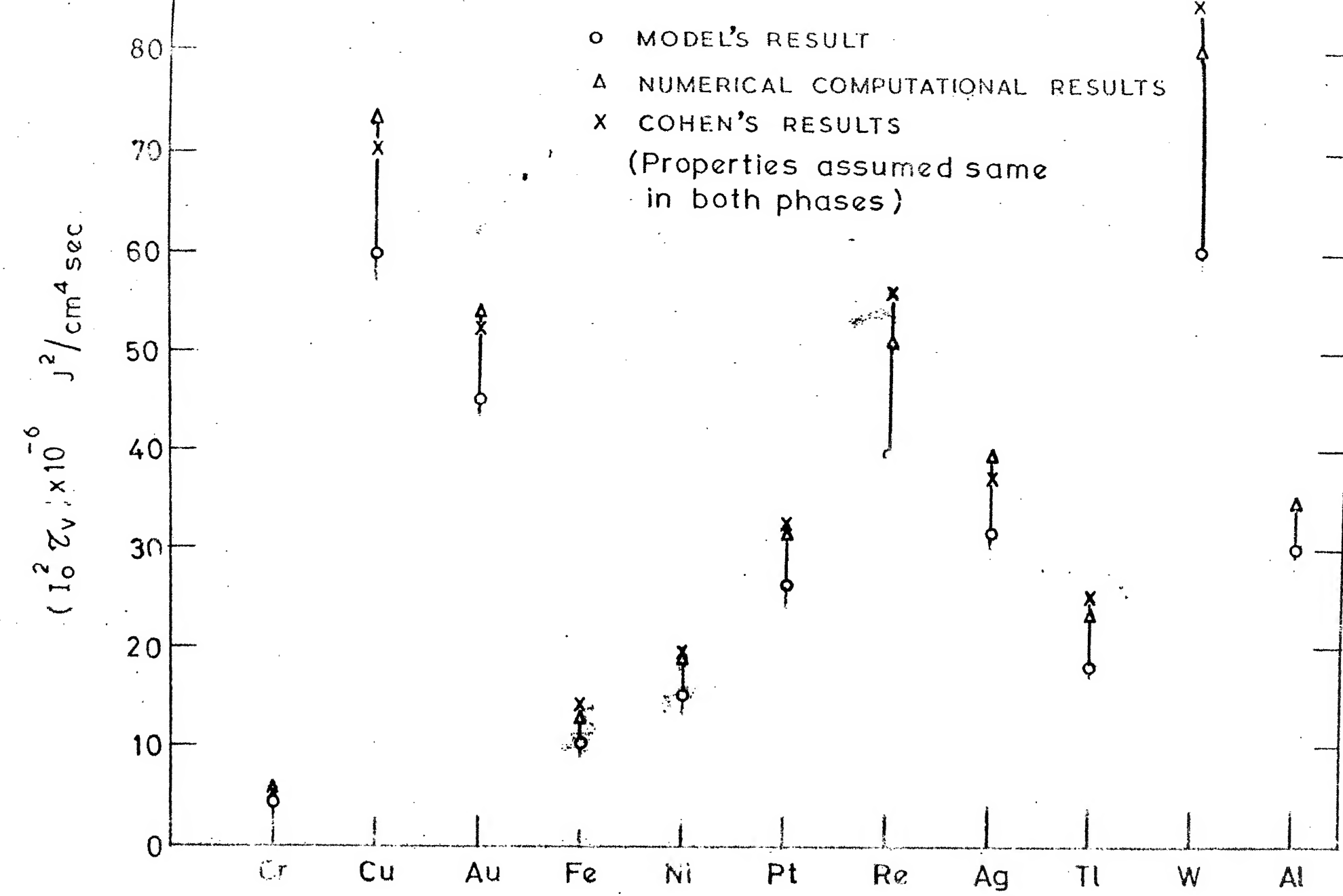
o MODEL'S RESULT
Δ NUMERICAL COMPUTATIONAL RESULTS
X COHEN'S RESULTS
(Properties assumed same in both phases)
$(I_o^2 \tau_v)_c \times 10^{-6}$ $J^2/cm^4 sec$
0
10
20
30
40
50
60
70
80
Cr
Cu
Au
Fe
Ni
Pt
Re
Ag
Tl
W
Al
Metarials

FIG. 2

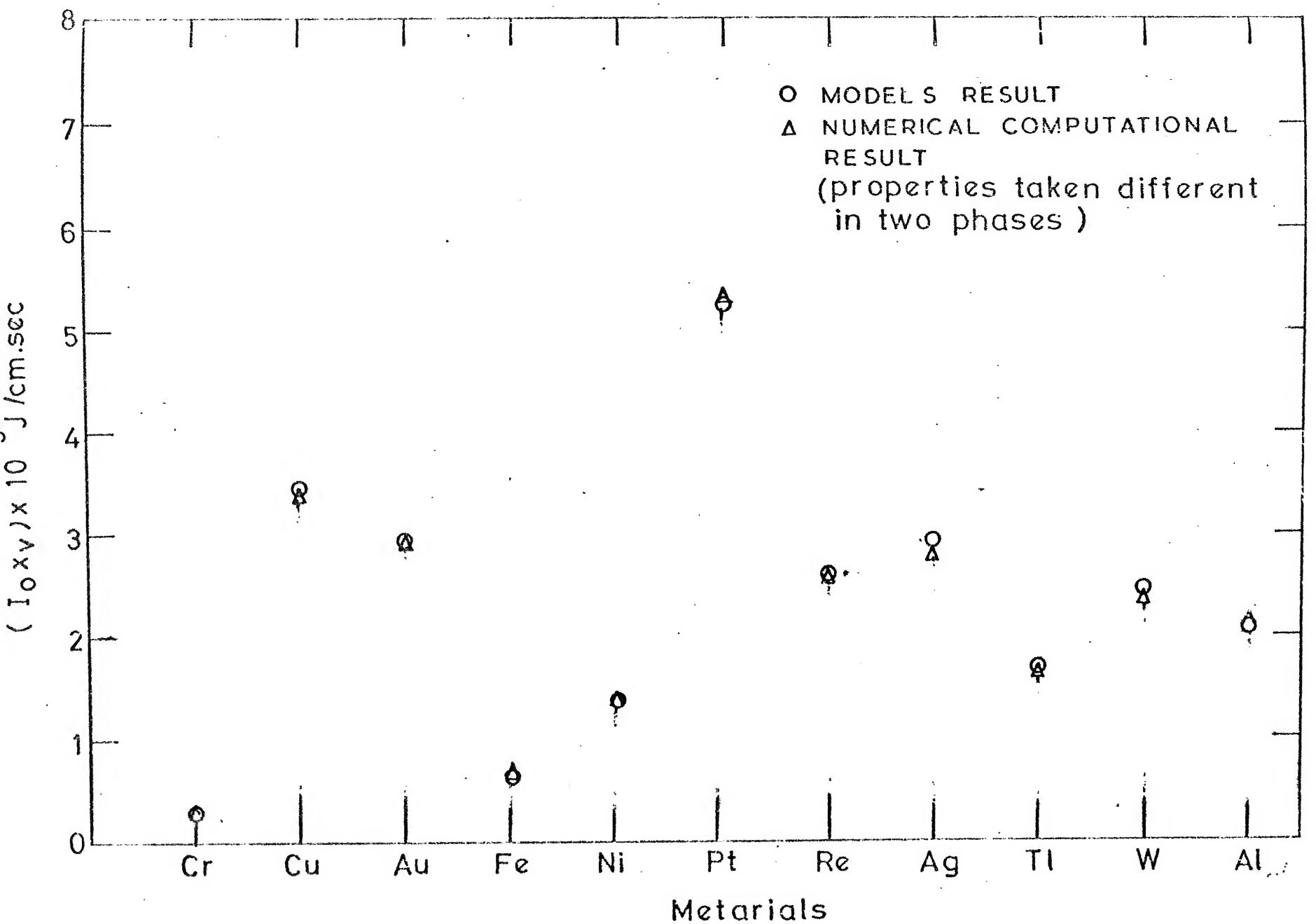
O MODEL S RESULT
Δ NUMERICAL COMPUTATIONAL RESULT
(properties taken different in two phases)
(I₀ x v) x 10⁻⁵ J /cm.sec
0
1
2
3
4
5
6
7
8
Cr
Cu
Au
Fe
Ni
Pt
Re
Ag
Tl
W
Al
Metarials

FIG. 4

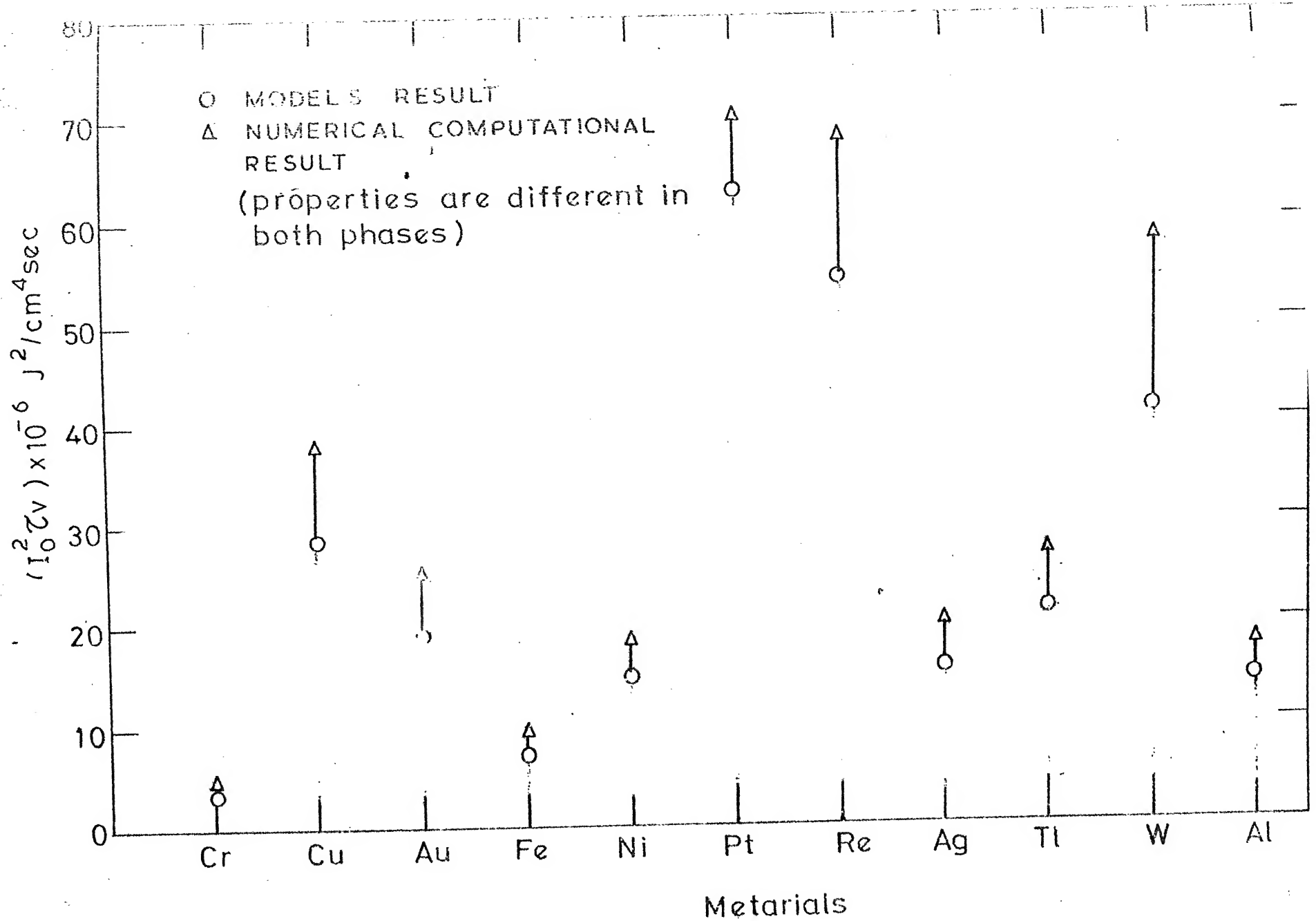
O MODELS RESULT
Δ NUMERICAL COMPUTATIONAL RESULT
(properties are different in both phases)
$(I_0^2\tau_v) \times 10^{-6}$ J^2/cm^{4}sec
80
70
60
50
40
30
20
10
0
Cr
Cu
Au
Fe
Ni
Pt
Re
Ag
Tl
W
Al
Metarials

FIG. 5

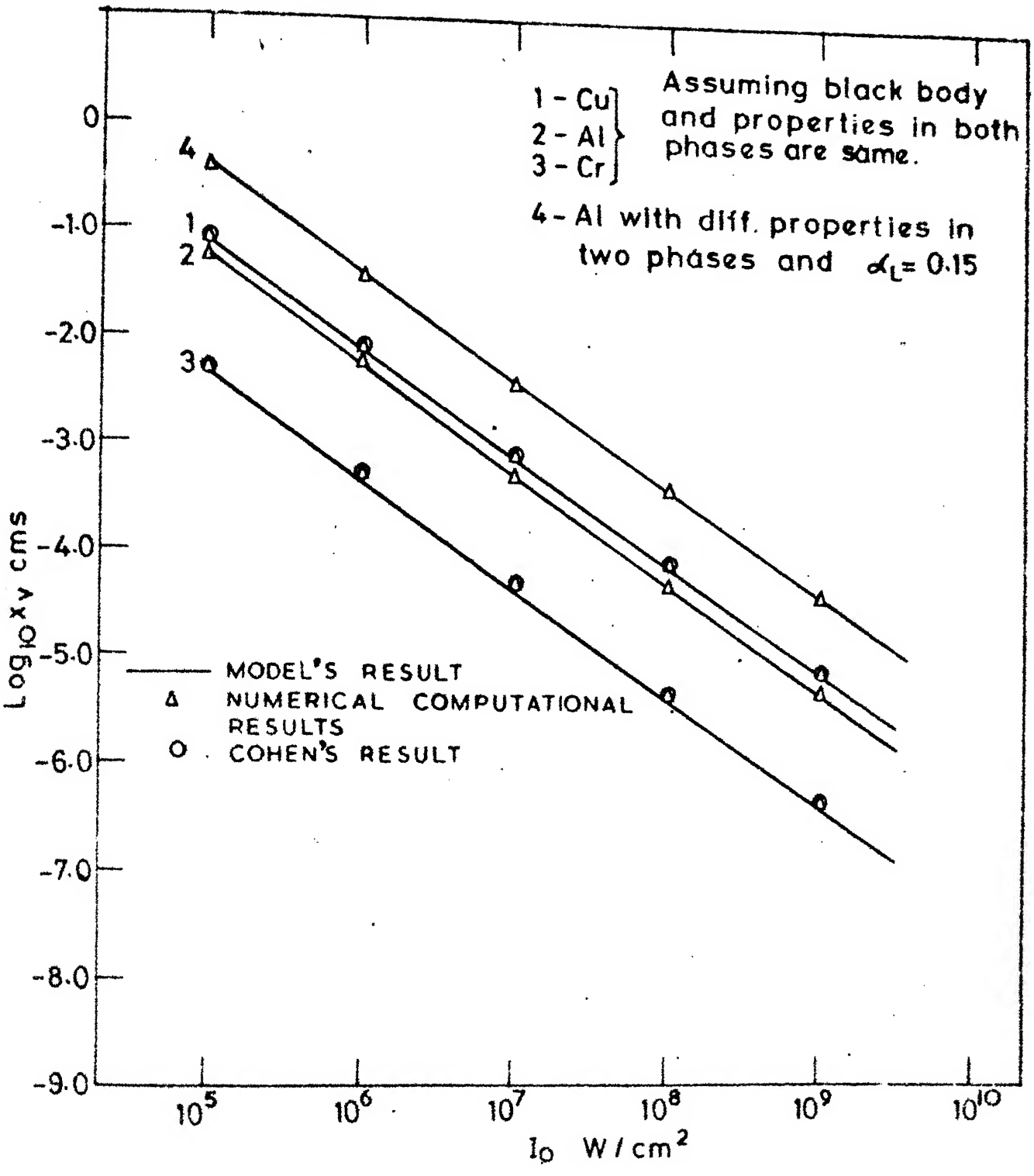
1 - Cu
2 - Al
3 - Cr
Assuming black body and properties in both phases are same.
4 - Al with diff. properties in two phases and $\alpha_L = 0.15$
MODEL'S RESULT
NUMERICAL COMPUTATIONAL RESULTS
COHEN'S RESULT
$\log_{10} x_v$ cms
I_0 W / cm^2
0
-1.0
-2.0
-3.0
-4.0
-5.0
-6.0
-7.0
-8.0
-9.0
10^5
10^6
10^7
10^8
10^9
10^{10}

FIG. 6

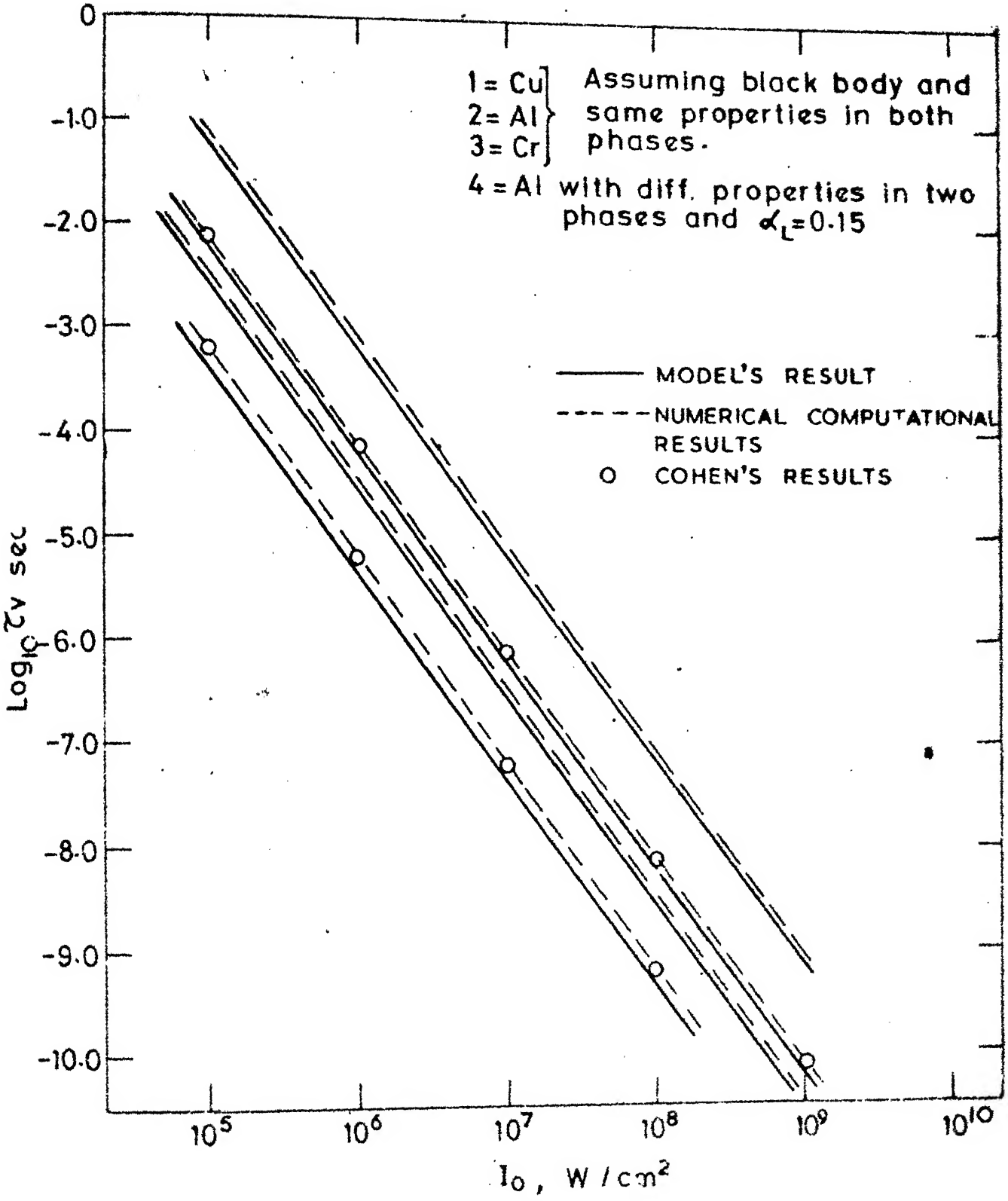
1 = Cu
2 = Al
3 = Cr
Assuming black body and same properties in both phases.
4 = Al with diff. properties in two phases and α_L=0.15
MODEL'S RESULT
NUMERICAL COMPUTATIONAL RESULTS
COHEN'S RESULTS
$\log_{10}\tau_v$ sec
I_o, W / cm^2
0
-1.0
-2.0
-3.0
-4.0
-5.0
-6.0
-7.0
-8.0
-9.0
-10.0
10^5
10^6
10^7
10^8
10^9
10^{10}

FIG. 7

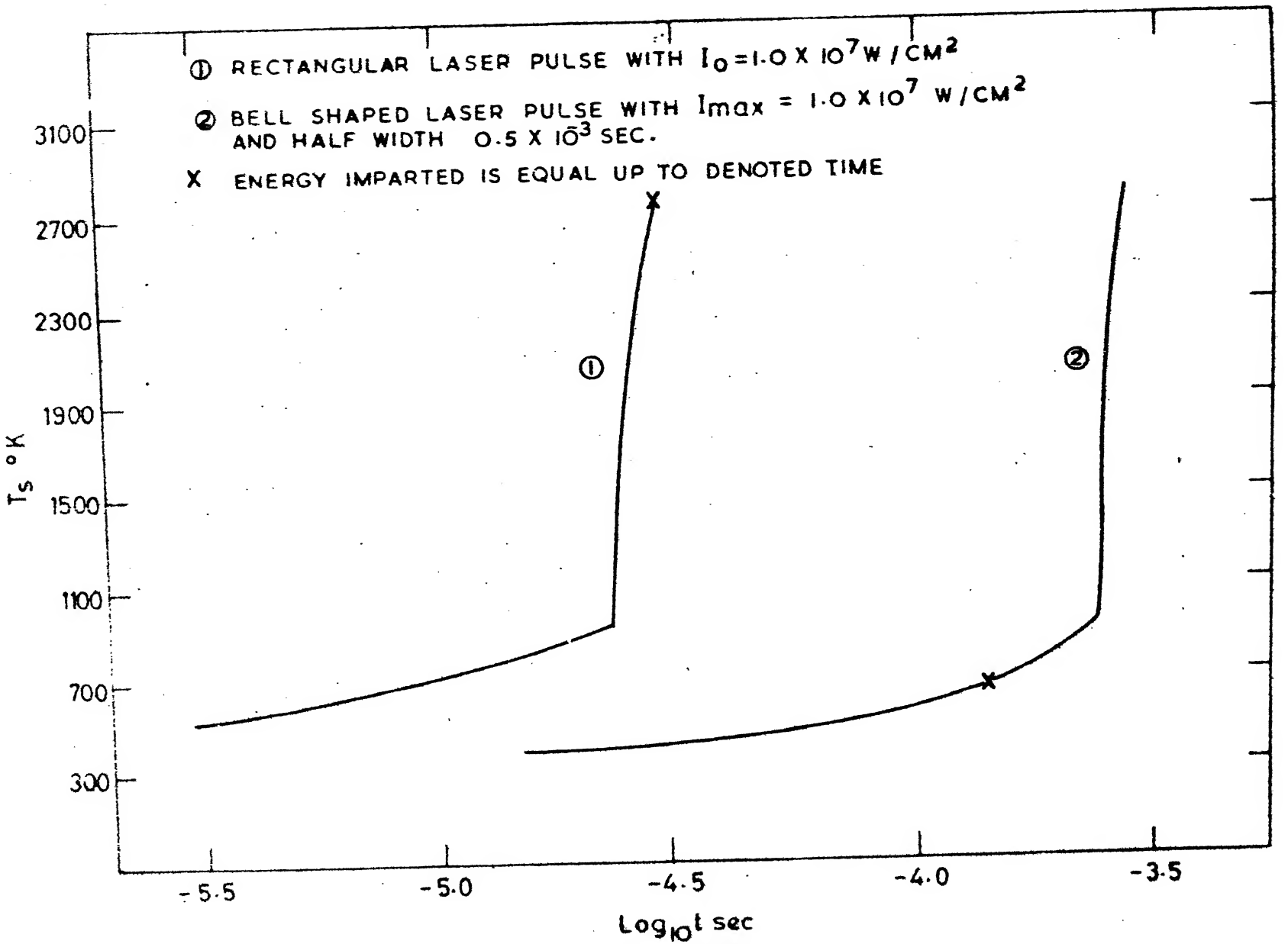
① RECTANGULAR LASER PULSE WITH $I_0 = 1.0 \times 10^7$ W/CM^2
② BELL SHAPED LASER PULSE WITH $I_{max} = 1.0 \times 10^7$ W/CM^2 AND HALF WIDTH 0.5×10^{-3} SEC.
X ENERGY IMPARTED IS EQUAL UP TO DENOTED TIME
3100
2700
2300
1900
1500
1100
700
300
T_s °K
①
②
-5.5
-5.0
-4.5
-4.0
-3.5
$\log_{10} t$ sec

FIG. 8

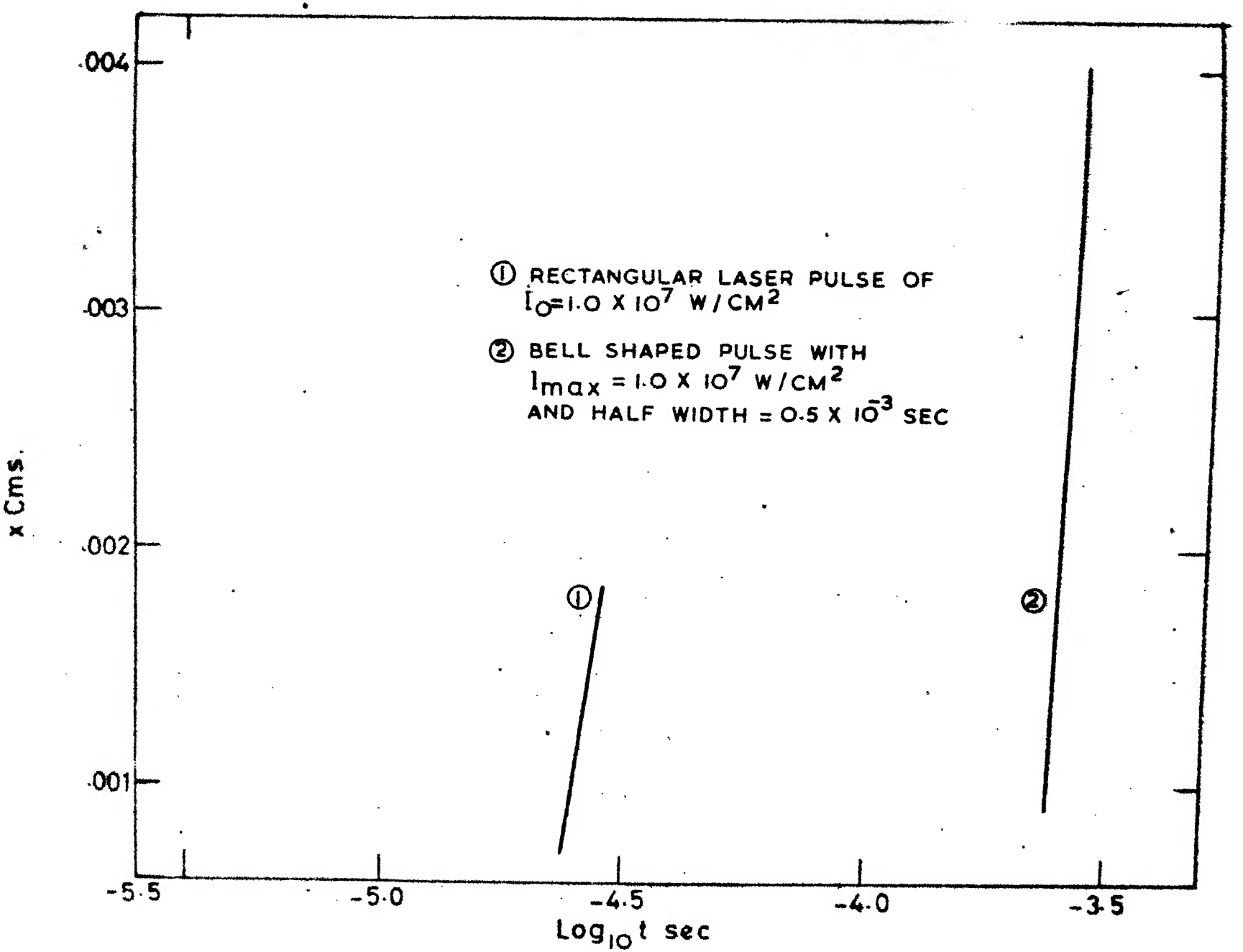
.004
.003
.002
.001
x Cms.
① RECTANGULAR LASER PULSE OF
$I_0 = 1.0 \times 10^7$ W/CM^2
② BELL SHAPED PULSE WITH
$I_{max} = 1.0 \times 10^7$ W/CM^2
AND HALF WIDTH = 0.5×10^{-3} SEC
①
②
-5.5
-5.0
-4.5
-4.0
-3.5
Log_{10} t sec

FIG. 9

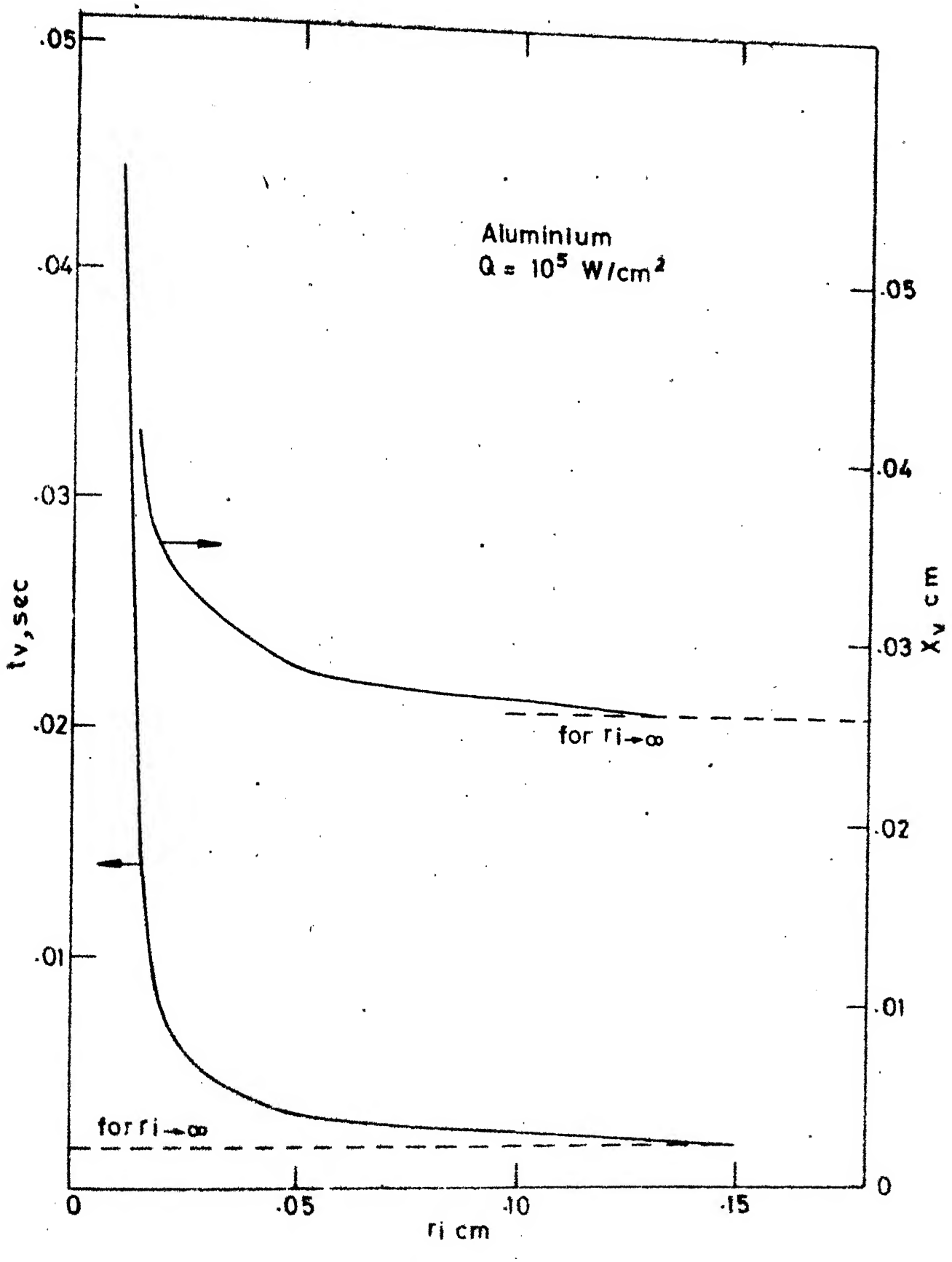
Aluminium
Q = 10^5 W/cm^2
for ri→∞
for ri→∞
tv, sec
xv cm
ri cm
.05
.04
.03
.02
.01
0
.05
.10
.15

FIG. 10

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```
C**********************************************************************
C**********************************************************************
C
C                    THESIS  PROGRAM
C                   ---------------------
C
C   This program computes the temperature distribution,the
C position of interface and interface velocity with respect
C to time.The body is one_dimensional finite or semi_infinite
C slab at a constant temperature TO initially.A prescribed
C heat flux is applied to its one face and other being insulated.
C The program is divided in three portions.First one calculates
C the temps. distribution in the slab untill the on_set of
C melting.Second one takes it as initial temp. distribution and
C computes temp.distribution,interface position and its velocity.
C To solve the heat conduction equation finite difference form
C with central diff. has been used.Third portion is a subroutine
C to solve simultaneous linear algebraic equations resulting
C from finite difference formulation with central difference.
C The material properties except density has been taken diff.
C in two phases but in a single phase they are const.with
C temp.
C-----------------------------------------------------------------------
C
C                    LIST OF VARIABLES
C                  ----------------------------
C AIO=Applied heat flux.  TO=Initial temperature.
C TM=Melting temp.          TB=Vaporisation temp.
C CONS,CONL are thermal conductivities in solid & liq. resp.
C CPS,CPL   are heat capacities in two phases.
C ABS,ABSL  are flux absorptivities in two phases.
C DIFS,DIFL are thermal diffusivities in two phases.
C DENS is density,taken same for both phases.
C ALM=Latent heat of fusion.
C AL=Slab width.
C AIMAX=Maximum flux in case of time varying flux.
C BO=Half width at half max.,flux variation is Gaussian.
C Program has been written in non_dimensional form.
C Properties values are in C.G.S.unit system.Temps.are in degree
C Kelvin.
C
C-----------------------------------------------------------------------
```

```
05100           DIMENSION TA(1000),T(1000),A(1000),B(1000),C(1000),D(1000),
05200           9XX(1000),DA(1000)
05300           AIO=1.0E+05;AIMAX=1.0E+07;BO=0.0005;TIMO=BO*SQRT(2.0)
05400           TO=300.0;ABS=1.0;ABSL=1.0
05500 C
05600           READ*,DENS,CONS,CPS,CONL,CPL,TM,TB,ALM,XI
05700 C
05800           N=1000;M=1000
05900           NM=N-1;NMM=N-2;NP=N+1
06000 C         ------------------------------------------------------------
06100 C          initialisation and nondimensionalisation of variables.
06200 C         ------------------------------------------------------------
06300           TIME=0.0;DIST=0.0;DT=0.000001
06400           ITER1=0;NIT=0;INT1=30;INT2=10
06500           ITER2=0
06600           AL=1.0
06700 C         ABS=0.03
06800 C         CONL=CONS;ABSL=ABS;DIFL=DIFS;CPL=CPS
06900 C         ABSL=0.15
07000           CONSL=CONS/CONL
07100           DX=AL/N
07200           DXH=DX*3.0/2.0
07300           DXT=DX*5.0/2.0
07400           DXN=DX/AL
07500           DIFS=CONS/(DENS*CPS)
07600           DIFL=CONL/(DENS*CPL)
07700           DTNS=DIFS*DT/(AL*AL)
07800           DTNL=DIFL*DT/(AL*AL)
07900           ALAMS=DTNS/(DXN*DXN)
08000           ALAML=DTNL/(DXN*DXN)
08100           ALMM=ALM+CPS*(TM-TO)
08200           VCM=CONL*(TB-TM)/(DENS*ALMM*AL)
08300           VC=CONL*(TB-TM)/(DENS*ALM*AL)
08400           Y=ALM/(CPS*(TM-TO))
08500           PRINT10,N,AIO,AL,DX,DT,Y
08600 10        FORMAT(5X,' N= ',I5,5X,'AIO= ',E15.8,5X,' AL= ',F4.1,5X,' DX
08700          8= ',F7.4,5X,' DT= ',E15.8,' Y= ',F5.2,//)
08800 C         ------------------------------------------------------------
08900 C          Solving diffusion eq. till the onset of melting.
09000 C         ------------------------------------------------------------
09100           DO20 I=1,N
09200           A(I)=0.0
09300           B(I)=0.0
09400           C(I)=0.0
09500           D(I)=0.0
09600 20        CONTINUE
09700           B(1)=1.0+ALAMS
09800           C(1)=-ALAMS
09900           DO30 I=2,NM
10000           A(I)=-ALAMS
10100           B(I)=1.0+2.0*ALAMS
10200 30        C(I)=-ALAMS
10300           A(N)=1.0
10400           B(N)=-1.0
```

```
10500           DO40 I=1,N
10600  40       T(I)=(TO-TM)/(TB-TM)
10700           DO110 JJ=2,M
10800           TIME=TIME+DT
10900           NJJ=JJ
11000           JJM=JJ-1
11100 C         ------------------------------------------------------------
11200 C         If flux is bell shaped,AIO is given as...
11300 C         AIO=AIMAX*EXP(-((TIME-TIMO)/BO)**2)
11400 C         ------------------------------------------------------------
11500           CO=DTNS*ABS*AIO*AL/((TB-TM)*CONS*DXN)
11600 C         PRINT45,CO
11700  45       FORMAT(5X,E15.8,/)
11800           D(1)=CO+T(1)
11900           DO50I=2,N-1
12000  50       D(I)=T(I)
12100           ITER1=ITER1+1
12200           CALL TRIDAG(A,B,C,D,N,XX)
12300           DO90 I=1,N
12400           T(I)=XX(I)
12500  90       TA(1)=T(I)*(TB-TM)+TM
12600 100       FORMAT(5X,F8.1,10X,E15.8)
12700           NIT=NIT+1
12800           TS=(3.0*T(1)-T(2))/2.0
12900           IF(NIT.LT.INT1) GO TO 101
13000           NIT=0
13100           TSA=TS*(TB-TM)+TM
13200           PRINT100,TSA,TIME
13300 101       CONTINUE
13400           IF(TS.GE.0.0) GO TO 130
13500 110       CONTINUE
13600           PRINT120
13700 120       FORMAT(10X,'Temp not reached to melting ',//)
13800           STOP
13900 130       CONTINUE
14000           PRINT132
14100 132       FORMAT(5X,'     TIME              TS                TN
14200          9EL               DIST ',//)
14250 C         ------------------------------------------------------------
14275 C          Melting has started.Calculations for liquid zone growth till
14287 C          three grid points are covered.
14293 C         ------------------------------------------------------------
14300           DT=DT*2.0
14400           DTNS=DIFS*DT/(AL*AL)
14500           DTNL=DIFL*DT/(AL*AL)
14600           ALAMS=DTNS/(DXN*DXN)
14700           ALAAL=DTNL/(DXN*DXN)
14800 C         CS=DT*ABS*AIO/(DENS*CPS*DX)
14900 C         CSN=DTNS*ABS*AIO*AL/((TB-TM)*CONS*DXN)
15000 C         CL=DT*ABSL*AIO/(DENS*CPL*DX)
15100 C         CLN=DTNL*ABSL*AIO*AL/((TB-TM)*CONL*DXN)
15200           PRINT135,TIME
15300 135       FORMAT(5X,' Time for melting ',E15.8,/)
15400 C         AIO=AIMAX*EXP(-((TIME-TIMO)/BO)**2)
15500           CLN=ABSL*AIO*AL/(CONL*(TB-TM))
15600           V=CLN+2.0*CONSL*(T(1)-TS)/DXN
```

```
15700           VA=V*VC
15800 C         VA=ABS*AIO/(DENS*ALMM)
15900           DIST=DIST+VA*DT
16000           DISTN=DIST/AL
16100           IF(DIST.GT.DX/2.0) GO TO 170
16200 140       DELX=DX/2.0-DIST
16300           DELXN=DELX/AL
16400           A(1)=0.0
16500           B(1)=1.0+DX/DELX
16600           C(1)=-1.0
16700           D(1)=0.0
16800           DO150 I=2,NM
16900           A(I)=-ALAMS
17000           B(I)=1.0+2.0*ALAMS
17100           C(I)=-ALAMS
17200 150       D(I)=T(I)
17300           A(N)=1.0
17400           B(N)=-1.0
17500           C(N)=0.0
17600           D(N)=0.0
17700           TIME=TIME+DT
17800           ITER2=ITER2+1;NIT=NIT+1
17900           CALL TRIDAG(A,B,C,D,N,XX)
18000           DO160 I=1,N
18100           T(I)=XX(I)
18200 160       TA(I)=T(I)*(TB-TM)+TM
18300           IF(NIT.LT.INT2) GO TO 161
18400           NIT=0
18500 C         PRINT100,TA(1),TIME
18600 161       CONTINUE
18700 C         AIO as below in case of bell shaped energy pulse.
18800 C         AIO=AIMAX*EXP(-((TIME-TIMO)/BO)**2)
18900           COIN=ABSL*AIO*AL/(CONL*(TB-TM))
19000           V=COIN+CONSL*T(1)/DELXN
19100           VA=V*VC
19200 C         ---------------------------------------------------------
19300 C          Interface velocity in model's case is as below.
19400 C         VA=ABS*AIO/(DENS*ALMM)
19500 C         ---------------------------------------------------------
19600           DIST=DIST+VA*DT
19700           TSA=(3.0*TA(1)-TA(2))/2.0
19800 C         PRINT165,TIME,TSA,TA(N),VA,DIST
19900 165       FORMAT(3X,E15.8,3X,F7.1,5X,F7.1,5X,E15.8,3X,E15.8,//)
20000           IF(DIST.GT.DX/2.0) GO TO 170
20100           GO TO 140
20200 170       IF(DIST.GT.DXN) GO TO 210
20300 180       DISTN=DIST/AL
20400           DELX=DIST-DX/2.0
20500           DELXN=DELX/AL
20600           TS=DISTN*COIN
20700           A(1)=0.0
20800           B(1)=1.0
20900           C(1)=0.0
21000           D(1)=DELXN*TS/DISTN
```

```
21100           A(2)=0.0
21200           B(2)=2.0-DELX/DX
21300           C(2)=DELX/DX-1.0
21400           D(2)=0.0
21500           DO190 I=3,NM
21600           A(I)=-ALAMS
21700           B(I)=1.0+2.0*ALAMS
21800           C(I)=-ALAMS
21900 190       D(I)=T(I)
22000           TIME=TIME+DT
22100           ITER2=ITER2+1;NIT=NIT+1
22200           CALL TRIDAG(A,B,C,D,N,XX)
22300           DO200 I=1,N
22400           T(I)=XX(I)
22500 200       TA(I)=T(I)*(TB-TM)+TM
22600 C         PRINT100,(TA(I),I=1,N)
22700           IF(NIT.LT.INT2) GO TO 201
22800           NIT=0
22900 C         PRINT100,TA(1),TIME
23000 201       CONTINUE
23100           V=T(1)/DELXN+CONSL*T(2)/(DXN-DELXN)
23200           VA=V*VC
23300 C         V=T(1)/DELXN
23400 C         VA=V*VCM
23500 C         PRINT165,VA,TIME
23600           DIST=DIST+VA*DT
23700           TSA=(3.0*TA(1)-TA(2))/2.0
23800 C         PRINT165,TIME,TSA,TA(N),VA,DIST
23900           IF(DIST.GT.DXH) GO TO 210
24000           GO TO 180
24100 210       IF(DIST.GT.DXT) GO TO 240
24200           DELX=DIST-DXH
24300           DELXN=DELX/AL
24400           A(1)=1.0
24500           B(1)=1.0+ALAML
24600           C(1)=-ALAML
24700 C         AIO=AIMAX*EXP(-((TIME-TIMO)/BO)**2)
24800           CLN=DTNL*ABSL*AIO*AL/((TB-TM)*CONL*DXN)
24900           D(1)=T(1)+CLN
25000           A(2)=1.0
25100           B(2)=-1.0-DX/DELX
25200           C(2)=0.0
25300           D(2)=0.0
25400           A(3)=0.0
25500           B(3)=2.0-DELX/DX
25600           C(3)=DELX/DX-1.0
25700           D(3)=0.0
25800           DO220 I=4,NM
25900           A(I)=-ALAMS
26000           B(I)=1.0+2.0*ALAMS
26100           C(I)=-ALAMS
26200 220       D(I)=T(I)
26300           TIME=TIME+DT
26400           ITER2=ITER2+1;NIT=NIT+1
26500           CALL TRIDAG(A,B,C,D,N,XX)
```

```
26600           DO230 I=1,N
26700           T(I)=XX(I)
26800  230      TA(I)=T(I)*(TB-TM)+TM
26900  C        PRINT100,(TA(I),I=1,N)
27000           IF(NIT.LT.INT2) GO TO 231
27100           NIT=0
27200  C        PRINT100,TA(1),TIME
27300  231      CONTINUE
27400           V=T(2)/DELXN+CONSL*T(3)/(DXN-DELXN)
27500           VA=V*VC
27600  C        V=T(2)/DELXN
27700  C        VA=V*VCM
27800  C        PRINT165,VA,TIME
27900           DIST=DIST+VA*DT
28000           TSA=(3.0*TA(1)-TA(2))/2.0
28100  C        PRINT165,TIME,TSA,TA(N),VA,DIST
28200           IF(DIST.GT.DXT) GO TO 240
28300           GO TO 210
28400  240      CONTINUE
28500  242      TIMM=((TM-TO)/(2.*ABS*AIO))**2*3.14*DENS*CPS*CONS
28600  C        TIMM gives analytical value of tm .
28700           PRINT260,TIMM
28800  260      FORMAT(5X,' TIMM=  ',E15.8,//)
28900  C        ------------------------------------------------------------
29000  C        Atleast three grid points have come in liquid region.
29100  C        Calculations hereafter.
29200  C        ------------------------------------------------------------
29300           NJJ=NJJ+1
29400           DO315 JJ=NJJ,M
29500           JJM=JJ-1
29600  C        PRINT280,DIST
29700  280      FORMAT(5X,' DIST= ',E15.8,//)
29800           Q=DIST/DX
29900           DO290 I=1,N
30000           NR=I
30100           NRMM=NR-2
30200           NRM=NR-1
30300           NRP=NR+1
30400           NRPP=NR+2
30500           IF(Q.LE.I) GO TO 300
30600  290      CONTINUE
30700           GO TO 500
30800  300      DELX=DIST-(NR*DX-DX/2.0)
30900           DELXN=DELX/AL
31000  C        PRINT310,NR
31100  310      FORMAT(5X,' NR= ',I10,/)
31200           DO320 I=1,NR
31300           A(I)=0.0
31400           B(I)=0.0
31500           C(I)=0.0
31600           D(I)=0.0
31700  320      CONTINUE
31800           B(1)=1.0+ALAML
31900           C(1)=-ALAML
32000  C        AIO=AIMAX*EXP(-((TIME-TIMO)/BO)**2)
32100           CLN=DTNL*ABSL*AIO*AL/((TB-TM)*CONL*DXN)
32200           D(1)=CLN+T(1)
```

```
32300           IF(DELX.LT.0.0) GO TO 350
32400           DO330 I=2,NRM
32500           A(I)=-ALAML
32600           B(I)=1.0+2.0*ALAML
32700           C(I)=-ALAML
32800  330      D(I)=T(I)
32900           A(NR)=1.0
33000           B(NR)=-1.0-DXN/DELXN
33100           C(NR)=0.0
33200           D(NR)=0.0
33300           A(NRP)=0.0
33400           B(NRP)=2.0-DELX/DX
33500           C(NRP)=DELX/DX-1.0
33600           D(NRP)=0.0
33700           DO340 I=NRPP,NM
33800           A(I)=-ALAMS
33900           B(I)=1.0+2.0*ALAMS
34000           C(I)=-ALAMS
34100  340      D(I)=T(I)
34200           A(N)=1.0
34300           B(N)=-1.0
34400           C(N)=0.0
34500           D(N)=0.0
34600           GO TO 380
34700  350      DO360 I=2,NRMM
34800           A(I)=-ALAML
34900           B(I)=1.0+2.0*ALAML
35000           C(I)=-ALAML
35100  360      D(I)=T(I)
35200           A(NRM)=1.0
35300           B(NRM)=-(2.0*DX+DELX)/(DX+DELX)
35400           C(NRM)=0.0
35500           D(NRM)=0.0
35600           A(NR)=0.0
35700           B(NR)=-1.0+DX/DELX
35800           C(NR)=1.0
35900           D(NR)=0.0
36000           DO370 I=NRP,NM
36100           A(I)=-ALAMS
36200           B(I)=1.0+2.0*ALAMS
36300           C(I)=-ALAMS
36400  370      D(I)=T(I)
36500           A(N)=1.0
36600           B(N)=-1.0
36700           C(N)=0.0
36800           D(N)=0.0
36900  380      CONTINUE
37000           DO390 I=1,N
37100  390      DA(I)=D(I)*(TH-TM)+TM
37200  C        PRINT400,(DA(I),I=1,N)
37300  400      FORMAT(5X,10(F10.1,3X),//,5X,10(F10.1,3X),//)
37400           ITER2=ITER2+1;NIT=NIT+1
37500           CALL TRIDAG(A,B,C,D,N,XX)
37600           TIME=TIME+DT
37700           DO420 I=1,N
37800           T(I)=XX(I)
```

```
37900 420      TA(I)=T(I)*(TB-TM)+TM
38000          PRINT430,(TA(I),I=1,NR)
38100 430      FORMAT(5X,10(F10.1,3X),/)
38200 C        PRINT440,T(1),T(NR),T(N)
38300 440      FORMAT(5X,'TEMPS= ',3(F10.2,3X),//)
38400          VN=(CONSL*T(NRP))/(DXN-DELXN)+T(NRM)/(DXN+DELXN)
38500 C        V=(CONS*(TA(NRP)-TM)/(DX-DELX)-CONL*(TM-TA(NRM))/(DX+DELX))/
38600 C        9(DENS*ALM)
38700          VA=VC*VN
38800          QL=-CONL*(TM-TA(NRM))/(DX+DELX)
38900          QS=-CONS*(TA(NRP)-TM)/(DX-DELX)
39000          QMELT=DENS*ALM*VA
39100 C        V=T(NRM)/(DXN+DELXN)
39200 C        VA=V*VCM
39300          DELX=VA*DT
39400          DELXN=DELX/AL
39500          TS=DXN*(T(1)-T(2))/DX+T(2)
39600          DIST=DIST+DELX
39700          TSA=(3.0*TA(1)-TA(2))/2.0
39800          IF(NIT.LT.INT2) GO TO 316
39900          NIT=0
40000 C        PRINT165,TIME,TSA,TA(N),VA,DIST
40100 316      CONTINUE
40200 C        PRINT165,TIME,TSA,TA(N),VA,DIST,QL,QS,QMELT
40300          IF(TS.GE.1.) GO TO 530
40400 315      CONTINUE
40500          PRINT560
40600 560      FORMAT(5X,' VAporisation not started ',//)
40700          GO TO 520
40800 500      PRINT490
40900 490      FORMAT(10X,'Whole length melted ',//)
41000 520      CONTINUE
41100 530      CONTINUE
41700          PRINT165,TIME,TSA,TA(N),VA,DIST
41800          PRINT600,ITER1,ITER2
41900 600      FORMAT(5X,' ITER1= ',I5,'  ITER2= ',I5,//)
42000          PRINT540
42100 540      FORMAT(5X,' BOILING STARTS ')
42200          STOP;END
```

```
42400 C     ------------------------------------------------------------
42500 CCC   Subroutine to solve tridiagonal matrix resulting from
42600 CCC   finite diff. formulation.
42700 C     ------------------------------------------------------------
42800       SUBROUTINE TRIDAG(A,B,C,D,N,XX)
42900       DIMENSION A(1000),B(1000),C(1000),D(1000),XX(1000),BETA(1000),
43000      9GAMA(1000)
43100       BETA(1)=B(1)
43200       GAMA(1)=D(1)/BETA(1)
43300       DO5 I=2,N
43400       IM=I-1
43500       BETA(I)=B(I)-A(I)*C(IM)/BETA(IM)
43600  5    GAMA(I)=(D(I)-A(I)*GAMA(IM))/BETA(I)
43700       XX(N)=GAMA(N)
43800       NM=N-1
43900       DO10 I=1,NM
44000       K=N-I
44100       KP=K+1
44200  10   XX(K)=GAMA(K)-C(K)*XX(KP)/BETA(K)
44300       RETURN
44400       END
```